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Finite Temperature and Density Effects in Higher Dimensions with and without Compactifications

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Abstract

Expressions for the thermodynamic potential of a Dirac fermion gas are represented at finite temperature with the chemical potential in an ultrastatic space $R^d \times S^N$. The high- and low- temperature expansions for the thermodynamic potential are obtained and, in particular, strongly degenerate fermi gas is investigated. For the Candelas-Weinberg model, sufficiently high "charge" density prevents the compactification of the extra space.

1 Introduction

These days many efforts have been made for pursuits of "the unification through higher dimensions." It is highly remarkable that supergravity theories have simple or unique structures in more than four dimensions.[1] Recent success in superstring theories suggests that higher dimensions are required not only for simplicity, but also the consistency of the theories.[2] Anyway, as far as we regard the extra dimensions as physical settings, we obviously need to comprehend the phenomena called "compactifications", in order to find out our four-dimensional world.

Recently, there appear many works investigating the behavior of the time dependence of the scale factor of extra spaces, that is, the "Kaluza-Klein cosmology."[3] Many people intend to explain the large amount of entropy in our universe in the same scenario. The scenario is referred to with a slogan "entropy comes from extra dimensions"[4] for this attempt. Therefore it is necessary to calculate the thermodynamic quantities in higher dimensional as well as curved spaces.

On the other hand, Actor derived the thermodynamic potential in arbitrary dimensions.[5] Particularly, we are interested in the case with non-vanishing chemical potentials. It is well known that symmetry restorations depend on finite density effects as well as finite temperature effects in four-dimensional universe.[6] Thus, one can expect that compactifications of the extra spaces are similarly influenced by finite density and temperature.[7, 8]

Since we want to know, at least, about the thermodynamic quantities before and after the compactifications, we have to obtain the thermodynamic potentials in curved-spaces. In the present paper, we derive the thermodynamic potential of a Dirac fermion field in curved spaces such as $T \times R^d \times S^N$.

The present paper is arranged as follows. In §2, we briefly review the thermodynamics and the effective potential in field theories with imaginary time formalism.[9] We treat only with a Dirac field throughout this paper. In §3, we show the high temperature expansion of the thermodynamic potentials in the space whose background metrics are that of $T \times R^d \times S^N$. The low temperature case is studied in §4. The strong]y degenerate femi gas in higher-dimensional and/or curved spaces are studied. In §5, we discuss the "Kaluza-Klein thermodynamics,"[10] especially in the case of non-vanishing chemical potentials. The last section is devoted to a summary and discussion.

2 The thermodynamic potential and the path integral

We introduce the thermodynamic potential Ω for an ensemble at temperature β^{-1} and chemical potential μ as follows:

$$Z_G = \exp(-\beta\Omega) = \mathrm{Tr}\ \exp\{-\beta(\hat{H} - \mu\hat{N})\}\,. \tag{1}$$

Here $\hat{H}$ is the Hamiltonian of the system and $\hat{N}$ is the particle number operator. Z_G is the so-called grand partition function. Equation (1) reads for fermions, after the mode expansion as

$$\begin{aligned} Z_G &= \exp(-\beta\Omega) \\ &= \prod_k [\exp(-\beta\omega_k)(1+\exp(-\beta(\omega_k-\mu)))(1+\exp(-\beta(\omega_k+\mu)))]\,, \end{aligned} \tag{2}$$

where the frequency of modes is given by $\omega_k = (\mathbf{k}^2 + M^2)^{1/2}$. Here the zero-point oscillation is included.

Now the entropy S, the pressure P and the particle number $\mathcal{N}$ for the system under consideration are given by the partial derivatives of $\Omega(\beta, V, \mu)$,

$$S = \beta^2 \frac{\partial\Omega}{\partial\beta}\,; \quad P = -\frac{\partial\Omega}{\partial V}\,; \quad \mathcal{N} = -\frac{\partial\Omega}{\partial\mu}\,. \tag{3}$$

The "particle number" introduced here represents "the particle number minus antiparticle number," which may be called "the charge asymmetry".

In the path integral language, the thermodynamic potential for a Dirac field with mass M is related to the one-loop effective potential,[5]

$$\ln \mathrm{Det}[\not{D} + M] = \mathrm{Tr} \ln[(\omega_n + i\mu)^2 + \omega_k^2] \quad \text{with} \quad \omega_k^2 = \mathbf{k}^2 + M^2\,, \tag{4}$$

where Tr means the integration and summation of all physical modes and degrees of freedom (i.e., including a trace of the Dirac matrix). Here, $\omega_n = \{2\pi/\beta\}(n+$

1/2), (n is an integer) is introduced in the imaginary time formalism for field theories at finite temperature.[9] The chemical potential in (4) can be regarded as the zeroth component of (imaginary) gauge field which is coupled with the charge density of the field.[5, 11]

We can find the relation between one-loop effective potential given by (4) and the "quantum mechanical"[8] expression of the thermodynamic potential (see (2)) through the following identities (for fermions):[12]

$$\begin{aligned}
&\sum_n \ln\left[\left(\frac{2\pi}{\beta}(n+1/2)+i\mu\right)^2+y^2\right] \\
&= \beta y + \ln(1+\exp\beta(\mu-y)) + \ln(1+\exp-\beta(\mu+y))\,. \qquad (5)
\end{aligned}$$

It is well known that the quantum vacuum energy at zero temperature comes from the first term on the right-hand side of (5). Apart from this zero-point energy, we derive the thermodynamic potential for a Dirac field in flat d-dimensional space using (5) (and expansions of the logarithms):[5]

$$\begin{aligned}
\Omega &= (\mathrm{tr}\,\mathbf{1})\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}\sum_{n=1}^{\infty}(-1)^n\cosh(\beta\mu n) \\
&\quad \times 2\left(\frac{2\beta M}{n}\right)^{(d+1)/2} K_{(d+1)/2}(\beta M n) \qquad (6)
\end{aligned}$$

where $K_\nu(x)$ denotes the modified Bessel function, V_d is the volume of d-dimensional space and $\mathrm{tr}\mathbf{1} = 2^{[(d+1)/2]}$.

We know another way of evaluating the one-loop effective potentials. In general, the one-loop quantum corrections contain divergences which need to be regularized. Using the zeta function regularization,[13] we obtain the following expressions for effective potentials:

$$\beta\Omega = \zeta'(0) + \ln(2\pi\mu_R^2)\zeta(0)\,,$$

where

$$\begin{aligned}
\zeta(s) &\equiv \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{V_d}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\int\frac{d^d\mathbf{k}}{(2\pi)^d} \\
&\quad \times \sum_{n=-\infty}^{\infty}\exp\left[-t\left\{\left(\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)+i\mu\right)^2+\mathbf{k}^2+M^2\right\}\right]\,, \qquad (7)
\end{aligned}$$

where μ_R^2 is a parameter which has the dimension of mass and comes from adjusting the scale of the measure of the path integral.[13] Very recently, Allen showed that μ_R^2 is only involved in the vacuum energy part of the thermodynamic potential which is independent of β.[14] Therefore, as far as we discard the vacuum energy, we do not need to worry about the regularization-scale μ_R^2 and the thermodynamic potential of the system can be derived from the first

term of (7). To see this, we use the following identity (see the Appendix):

$$\sum_{n=-\infty}^{\infty} \exp\left\{-t\left(\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)+i\mu\right)^2\right\}$$
$$= \frac{\beta}{(4\pi)^{1/2}} t^{-1/2}\left[1+2\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\exp\left(-\frac{\beta^2 n^2}{4t}\right)\right]. \quad (8)$$

Then we can divide $\zeta(s)$ into two parts:

$$\begin{aligned}
\zeta(s) &= \zeta_0(s)+\zeta_\beta(s)\,, \\
\zeta_0(s) &= \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{\beta}{(4\pi)^{1/2}}\frac{V_d}{\Gamma(s)}\int_0^\infty dt\, t^{s-3/2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}\exp\left\{-t\left(\mathbf{k}^2+M^2\right)\right\}\,, \\
\zeta_\beta(s) &= \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{\beta}{(4\pi)^{1/2}}\frac{V_d}{\Gamma(s)}\int_0^\infty dt\, t^{s-3/2}\int\frac{d^d\mathbf{k}}{(2\pi)^d}2\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu) \\
&\quad\times\exp\left\{-t\left(\mathbf{k}^2+M^2\right)-\frac{\beta^2n^2}{4t}\right\}. \qquad (9)
\end{aligned}$$

$\zeta_0'(0)$ contributes to the thermodynamic potential as the temperature-independent vacuum energy. Thus, we only consider the contribution from $\zeta_\beta'(0)$. In fact, $\Gamma(s)\zeta_\beta(s)$ does not diverge when $s\to 0$ and $\Gamma(s)\sim s^{-1}$, so we can easily find:

$$\begin{aligned}
\Omega &= \frac{1}{\beta}\zeta_\beta'(0) \\
&= \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{V_d}{(4\pi)^{1/2}}\int_0^\infty dt\, t^{-3/2}\int\frac{d^d\mathbf{k}}{(2\pi)^d} \\
&\quad\times 2\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu)\exp\left\{-t\left(\mathbf{k}^2+M^2\right)-\frac{\beta^2n^2}{4t}\right\} \\
&= \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{V_d}{(4\pi)^{(d+1)/2}}\int_0^\infty dt\, t^{-(d+1)/2-1} \\
&\quad\times 2\sum_{n=0}^{\infty}(-1)^n\cosh(n\beta\mu)\exp\left\{-tM^2-\frac{\beta^2n^2}{4t}\right\} \\
&= \mathrm{tr}\,\mathbf{1}\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}\sum_{n=1}^{\infty}(-1)^n\cosh(n\beta\mu) \\
&\quad\times 2\left(\frac{2\beta M}{n}\right)^{(d+1)/2} K_{(d+1)/2}(\beta Mn)\,. \qquad (10)
\end{aligned}$$

This coincides with (6).

3 The high-temperature expansion

In this section, let us derive the thermodynamic potential in curved space. We consider the static backgrouud metric of $R^d\times S^N$ as a simple but sufficiently

general case. In order to treat this case, we generalize (10) as follows:

$$\begin{aligned}\Omega &= \frac{\mathrm{tr}\,\mathbf{1}}{2}\frac{V_d}{(4\pi)^{(d+1)/2}}\int_0^\infty dt\, t^{-(d+1)/2-1}2\sum_{n=1}^\infty(-1)^n\cosh(n\beta\mu) \\ &\quad\times\sum_{l=0}^\infty(2d_l)\exp\left\{-t(\omega_l^2+M^2)-\frac{\beta^2n^2}{4t}\right\}, \end{aligned}\tag{11}$$

where

$$d_l=\frac{\Gamma(N+l)}{l!\Gamma(N)},\quad \omega_l=\left(l+\frac{N}{2}\right)\frac{1}{a}$$

and $\mathrm{tr}\mathbf{1}=2^{[D/2]}$ with $D=d+N+1$. a is the radius of the hypersphere S^N.

Now, we consider high-temperature case such as $\beta/a\ll 1$. In this case, we can use the following formula for infinite sum related with theta function to expand (11) with respect to β/a:

$$\begin{aligned}&\sum_{l=0}^\infty d_l\exp\left\{-\left(l+\frac{N}{2}\right)^2x\right\} \\ &\qquad=\frac{1}{2}\frac{\Gamma(N/2)}{\Gamma(N)}x^{N/2}\left[1-\frac{1}{12}N(N-1)x+O(x^2)\right].\end{aligned}\tag{12}$$

This identity is explained in terms of theta function. (See the Appendix.) Making use of Eq. (12), we can expand Eq. (11) as

$$\begin{aligned}\Omega &= (\mathrm{tr}\,\mathbf{1})\frac{V_dV_N}{(4\pi)^{(d+N+1)/2}}\beta^{-(d+N+1)}\sum_{n=1}^\infty(-1)^n\cosh(n\beta\mu) \\ &\quad\times\left[2\left(\frac{2\beta M}{n}\right)^{(d+N+1)/2}K_{(d+N+1)/2}(\beta Mn)\right. \\ &\quad\left.-2\frac{N(N-1)}{12}\frac{\beta^2}{a^2}\left(\frac{2\beta M}{n}\right)^{(d+N-1)/2}K_{(d+N-1)/2}(\beta Mn)+\cdots\right],\end{aligned}\tag{13}$$

where $V_N=2\pi^{(N+1)/2}a^N/\Gamma((N+1)/2)$ is the volume of S^N. This first term on r.h.s. of Eq. (13) is the thermodynamic potential in flat $(d+N)$-dimensional space, while the second and further terms correspond to the deviation from the flat-space case.

When $\mu=0$, one can and the same expression as Dowker's.[15] For even $D(=1+d+N)$, the summation over n becomes a polynomial (i.e., finite terms).[5] We show here the expression in the simple massless case for later convenience:

$$\begin{aligned}\Omega &= (\mathrm{tr}\,\mathbf{1})\frac{V_dV_N}{(4\pi)^{(d+N+1)/2}}\beta^{-(d+N+1)} \\ &\quad\times\left[2^{d+N+1}\Gamma\left(\frac{d+N+1}{2}\right)\sum_{n=1}^\infty(-1)^n\frac{\cosh(n\beta\mu)}{n^{d+N+1}}\right. \\ &\quad\left.-2^{d+N-1}\Gamma\left(\frac{d+N-1}{2}\right)\frac{N(N-1)}{12}\frac{\beta^2}{a^2}\sum_{n=1}^\infty(-1)^n\frac{\cosh(n\beta\mu)}{n^{d+N-1}}+\cdots\right]\end{aligned}\tag{14}$$

4 The low temperature expansion and the strongly degenerate Fermi gas

Actor gave the low temperature expansion of the thermodynamic potential.[5] However, the low-temperature approximation becomes meaningless when $M^2 < \mu^2$. When $M^2 < \mu^2$, it is well known that Femi gas degenerates strongly at low temperature. Fortunately, we know the formula for the polylogarithmic functions $\mathrm{Li}_N(x) = \sum_{n=1}^{\infty} x^n/n^N$:[16]

$$\begin{aligned}\mathrm{Li}(-y^{-1}) &= (-1)^{N-1}\mathrm{Li}_N(-y) \\ &\quad -\sum_{r=0}^{N-2}\frac{(-1)^r}{r!}(\ln y)^r[1-(-1)^{N-1-r}](1-2^{r-N+1})\zeta(N-r) \\ &\quad -\frac{(-1)^N}{N!}(\ln y)^N\,, \qquad (15)\end{aligned}$$

and this gives another method for expanding Ω at low temperature. Hereafter, we consider $M = 0$ case, for simplicity. The generalization to the massive case is straightforward.

In d-dimensional flat space (which means dimension of space-time is $1 + d$), the thermodynamic potential for a massless Dirac field is given by (cf. (14))

$$\begin{aligned}\Omega &= (\mathrm{tr}\mathbf{1})\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}2^{d+1}\Gamma\left(\frac{d+1}{2}\right)\sum_{n=1}^{\infty}(-1)^n\frac{\cosh(n\beta\mu)}{n^{d+1}} \\ &= \left(\frac{\mathrm{tr}\mathbf{1}}{2}\right)\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}2^{d+1}\Gamma\left(\frac{d+1}{2}\right)[\mathrm{Li}_{d+1}(-e^{\beta\mu})+\mathrm{Li}_{d+1}(-e^{-\beta\mu})] \qquad (16)\end{aligned}$$

Applying (15) to (16), we can reexpress it as

$$\begin{aligned}\Omega &= \frac{\mathrm{tr}\mathbf{1}}{2}\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}2^{d+1}\Gamma\left(\frac{d+1}{2}\right) \\ &\quad\times\left[\frac{-(\beta\mu)^{d+1}}{(d+1)!}\left\{1+(d+1)!\sum_{b=1}^{[(d+1)/2]}2\zeta(2b)\frac{(\beta\mu)^{-2b}}{(d+1-2b)!}(1-2^{1-2b})\right\}\right. \\ &\quad\left.+\{1+(-1)^d\}\mathrm{Li}_{d+1}(-e^{-\beta\mu})\right]\,. \qquad (17)\end{aligned}$$

It is apparent that this expression enables us to approximate itself at low temperature by evaluating the summation and the polylogarithmic functions appropriately.

Furthermore, if $d + 1$ is even, we get the exact form of the thermodynamic potential which is expressed as polynomials:

$$\begin{aligned}\Omega &= -\frac{\mathrm{tr}\mathbf{1}}{2}\frac{V_d}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)}\frac{\mu^{d+1}}{d+1} \\ &\quad\times\left[1+(d+1)!\sum_{b=1}^{(d+1)/2}\frac{(\beta\mu)^{-2b}}{(d+1-2b)!}2(1-2^{1-2b})\zeta(2b)\right]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mathrm{tr}\mathbf{1}}{2}\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}2^{d+1}\Gamma\left(\frac{d+1}{2}\right) \\
&\quad\times\left[\sum_{c=0}^{(d-1)/2}\frac{(\beta\mu)^{2c}}{(2c)!}2\zeta(d+1-2c)(1-2^{2c-d})+\frac{(\beta\mu)^{d+1}}{(d+1)!}\right]. \qquad (18)
\end{aligned}
$$

This is exactly the same expression as the one obtained by the high-temperature expansion.[5]

Let us turn our attention to the space $R^d \times S^N$ again. Integration over t in (11) yields when $M = 0$, (cf. (10))

$$
\begin{aligned}
\Omega &= (\mathrm{tr}\mathbf{1})\frac{V_d}{(4\pi)^{(d+1)/2}}\beta^{-(d+1)}\sum_{n=1}^{\infty}(-1)^n\cosh(\beta\mu n) \\
&\quad\times\sum_{l=0}^{\infty}(2d_l)2\left(\frac{2\beta\omega_l}{n}\right)^{(d+1)/2}K_{(d+1)/2}(\beta\omega_l n)\,. \qquad (19)
\end{aligned}
$$

Using the integral representation

$$
K_\nu(z)=\frac{\sqrt{\pi}(z/2)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zx}(x^2-1)^{\nu-1/2}dx\,, \qquad (20)
$$

the sum over n can be performed to give

$$
\begin{aligned}
\Omega &= -(\mathrm{tr}\mathbf{1})\frac{V_d}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)}\sum_l d_l\omega_l^{d+1} \\
&\quad\times\int_1^\infty dx\,(x^2-1)^{d/2}\left\{\frac{1}{e^{\beta(\omega_l x-\mu)+1}}+(\mu\to-\mu)\right\}. \qquad (21)
\end{aligned}
$$

Now we can analyze the low-temperature case by using the method which can be found in many textbooks.[17]

First of all, we examine the zero-temperature ($\beta\to\infty$) case. Note, in the limit $\beta\to\infty$,

$$
\frac{1}{e^{\beta x}+1}\overset{\beta\to\infty}{\longrightarrow}\theta(-x)\,, \qquad (22)
$$

where $\theta(x)$ is the step function. The expression (21) reduces to the following form by means of (22) at zero temperature:

$$
\Omega=-\mathrm{tr}\mathbf{1}\frac{V_d}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)}\sum_{l=0}^{l_m}d_l\omega_l^{d+1}\int_1^{\mu/\omega_l}(x^2-1)^{d/2}dx\,, \qquad (23)
$$

where l_m is the largest integer satisfying $\omega_{l_m}<\mu$ and if $\omega_0>\mu$, then $\Omega=0$.

As a check on the efficiency of this approach, let us consider the limit $a\to\infty$. In this situation, the sum over l is reduced to the integration by the substitutions:

$$
\omega_l\to z\,,\quad \sum_{l=0}^{l_m}d_l\to\frac{a^N}{\Gamma(N)}\int_0^\mu dz\,z^{N-1}\,. \qquad (24)
$$

Consequently, we can show:

$$\begin{aligned}
\Omega &\sim -\mathrm{tr}\mathbf{1}\frac{V_d}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)}\frac{a^N}{\Gamma(N)}\int_0^\mu dz\, z^{N+d}\int_1^{\mu/z}dx\,(x^2-1)^{d/2} \\
&= -\mathrm{tr}\mathbf{1}\frac{V_dV_N}{(4\pi)^{(d+N)/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{N}{2}\right)}\frac{\mu^{d+N+1}}{d+N+1}\int_0^1 dy\, y^{N-1}(1-y^2)^{d/2} \\
&= -\frac{\mathrm{tr}\mathbf{1}}{2}\frac{V_dV_N}{(4\pi)^{(d+N)/2}}\frac{1}{\Gamma\left(\frac{d+N+2}{2}\right)}\frac{\mu^{d+N+1}}{d+N+1}\,. \qquad (25)
\end{aligned}$$

As is expected, this result corresponds to the one obtained from (17) in flat $(d+N)$-dimensional space at zero temperature.

Particularly, we are interested in the case $d=3$. In this case, one finds:

$$\begin{aligned}
\Omega &= -4\frac{V_3}{24\pi^2}\sum_{l=0}^{l_m}d_l\left[\mu(\mu^2-\omega_l^2)^{1/2}\left(\mu^2-\frac{5}{2}\omega_l^2\right)\right. \\
&\qquad\left.+\frac{3}{2}\omega_l^4\ln\left\{\frac{\mu}{\omega_l}+\left(\frac{\mu^2}{\omega_l^2}-1\right)^{1/2}\right\}\right]\,. \qquad (26)
\end{aligned}$$

From this expression, we realize that (26) is merely the sum of the thermodynamic potential in four-dimensional space-time at zero temperature [18] over the discrete mass levels which arise from compactification of N dimensions. The higher-modes than the chemical potential are frozen out. This interpretation is easily understood by drawing the figure which is analogous to the one in the paper by Barr and Brown,[4] but where we replace T with μ. Of course, in our case, the extra space S^N does not admit zero modes, therefore the situation is slightly different. If we consider the case that the extra-space has zero modes, we will obtain the exact expression for Ω in the flat d-dimensiollal space when μ is smaller than the first non-zero massive mode.

Before considering finite but still low temperature effect we show another example, the thermodynamic potential in S^N(i.e., $d=0$),

$$\begin{aligned}
\Omega &= -(\mathrm{tr}\mathbf{1})\sum_{l=0}^{l_m}d_l(\mu-\omega_l) \\
&= -(\mathrm{tr}\mathbf{1})\frac{(l_m+N)!}{l_m!(N-1)!}\left(\frac{\mu a}{N}-\frac{l_m+(N+1)/2}{N+1}\right)\frac{1}{a}\,. \qquad (27)
\end{aligned}$$

where

$$l_m=\left[\mu a-\frac{N}{2}\right]\,.$$

By using relationship (3), we derive the particle number

$$\mathcal{N}=-\frac{\partial\Omega}{\partial\mu}=(\mathrm{tr}\mathbf{1})\frac{(l_m+N)!}{l_m!N!}\,. \qquad (28)$$

We can see the effect of the discreteness of the Kaluza-Klein mass level from (28). One can also find self-consistent solutions of Einstein equations after balancing between the stress tensor of the degenerate Fermi gas with the Casimir effect and the cosmological constant as the case at finite temperature considered by Dowker et al.[19] The cosmological constant in this case plays an analogous role to the bag constant in MIT bag model.[18] Although the above example may be an interesting exercise, we do not study this situation in this paper.

Now, let us return to treating low-temperature effect applying the textbook-formula [17] to (21), we can get the following low-temperature expansion for Ω:

$$\begin{aligned}\Omega = \quad - \quad & \mathrm{tr}\mathbf{1}\frac{V_d}{(4\pi)^{d/2}}\frac{1}{\Gamma\left(\frac{d+2}{2}\right)}\sum_{l=0}^{l_m} d_l\left[\omega_l^{d+1}\int_1^{\mu/\omega_l} dy\,(y^2-1)^{d/2}\right.\\ & +\frac{1}{\beta^2}\frac{\pi^2}{6}d\cdot\mu(\mu^2-\omega_l^2)^{d/2-1}+\frac{1}{\beta^4}\frac{7\pi^4}{3\times 5!}d(d-2)\\ & \left.\times\mu\{(d-1)\mu^2-3\omega_l^2\}(\mu^2-\omega_l^2)^{d/2-3}+\cdots\right]. \end{aligned} \tag{29}$$

For the manifold which has zero-modes in contrast with S^N of our case, one can find that the expression for the thermodynamic potential agrees with that in flat d-dimensional space derived by (17) when $0 < \mu <$ (the mass of the lowest massive mode).

5 Kaluza-Klein thermodynamics and instability at finite density

In this section, We first examine the themodynamics in the space $R^d \times S^N$. Simllar case with vanishing chemical potential is extensively investigated by Tosa,[10] so we will concentrate our attention to the case at finite density. The remainder of this section is devoted to the investigation of instabilities of compactifications induced by quantum effects [20] at finite density.

Kaluza-Klein thermodynamics at finite density

We consider a massless Dirac field in the space $R^d \times S^N$. under such circumstances, we assume the thermodmamic potential Ω must be of the form

$$\Omega = \frac{1}{\beta}\left(\frac{R}{a}\right)^d f(\mu a, \beta/a) \equiv \frac{1}{\beta}\left(\frac{R}{a}\right)^d f(x,y)\,, \tag{30}$$

where R is the scale factor of the flat d-dimensional space and a is the radius of the N-dimensional hypersphere S^N.

The chemical potential is introduced for a fermion field in $(d+N)$-dimensional space. In the Kaluza-Klein sense, we find indefinitely many particles in d-dimensional space, however, the particle number defined in this paper is to be conserved, because this may be regarded as the charge asymmetry in the system.

Now let us derive several thermodynamic quantities using Eq. (30). We obtain

$$\begin{aligned}
PV_dV_N &= -\frac{1}{d}R\frac{\partial\Omega}{\partial R} = -\frac{1}{\beta}\left(\frac{R}{a}\right)^d f\,, \\
QV_dV_N &= -\frac{1}{N}a\frac{\partial\Omega}{\partial a} = -\frac{1}{N}\frac{1}{\beta}\left(\frac{R}{a}\right)^d\left[-d\cdot f + x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]\,, \\
S &= \beta^2\frac{\partial\Omega}{\partial\beta} = \left(\frac{R}{a}\right)^d\left[-f + y\frac{\partial f}{\partial y}\right]\,, \\
\mu\mathcal{N} &= -\mu\frac{\partial\Omega}{\partial\mu} = -\frac{1}{\beta}\left(\frac{R}{a}\right)^d x\frac{\partial f}{\partial x}\,, \\
E &= \Omega + \mu\mathcal{N} + \frac{1}{\beta}S = \frac{1}{\beta}\left(\frac{R}{a}\right)^d\left[-x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right]\,, \qquad (31)
\end{aligned}$$

where V_d, V_N, P, Q, S, $\mathcal{N}$ and E are, the volume of R^d, the volume of S^N, the pressue for R^d, the pressure for S^N, the entropy, the particle number and the internal energy, respectively. We notice immediately,

$$-\rho + d\cdot P + N\cdot Q = 0\,, \quad \text{here} \quad \rho = \frac{E}{V_dV_N}\,. \qquad (32)$$

This relation has often been mentioned, and is equivalent to state the thermal contribution to the stress tensor is traceless.

On the other hand, the conservation of the stress tensor gives

$$dE + PV_NdV_d + QV_ddV_N = TdS + \mu d\mathcal{N} = 0\,. \qquad (33)$$

It is natural to assume the entropy and the particle number are conserved separately. This point will be discussed again in the next section.

Instability of compactification at finite density

In order to gain stable compactifications of extra spaces, various mechanisms are proposed. Candelas and Weinberg [20] considered quantum effects of matter fields in the case of the compact spaces, S^N. Many other authors showed the extended versions of their model and the presence of various sorts of instabilities.[21, 22] Recently, Accetta and Kolb exhibited the finite temperature instability for compactifications, and the critical temperature for the instability.[23] Here, we show the finite density instability, which can be discussed almost parallel to the work by Accetta and Kolb.

At high density ($\beta\mu \gg 1$), the themodynamic potential for Dirac fields can be written in a form

$$\Omega = \frac{V_d}{a^{d+1}}(C_N - C(\mu a)^{d+N+1})\,. \qquad (34)$$

Note here Ω contains the Casimir stress energy which is determined at one-loop level when the space-time dimensionality is odd. The pressure for the internal space, Q, and the particle number $\mathcal{N}$ are computed from (34):

$$\begin{aligned} Q &= -\frac{1}{V_N}\frac{1}{N}a\frac{\partial\Omega}{\partial a} \\ &= \frac{V_d}{V_N}\frac{1}{a^{d+1}}[(d+1)C_N + N\cdot C(\mu a)^{d+N+1}]\,, \\ \mathcal{N} &= -\frac{\partial\Omega}{\partial\mu} \\ &= \frac{V_d}{a^d}C(d+N+1)(\mu a)^{d+N}\,. \end{aligned} \tag{35}$$

We recognize the similarity to the high-temperature case.[22] The only difference is the exchange of μ and T. Therefore we can immediately obtain the critical value for μ:

$$\mu_{crit} \sim \frac{1}{a_0}\left[\frac{C_N}{C}\frac{2(d-1)}{N}\left\{\frac{d+1}{N}+1\right\}\right]^{1/(N+d+1)}\,, \tag{36}$$

where a_0 is the static radius of S^N. Using this, we conclude that there are no stable compactifications when $\mathcal{N}/V_d > (\mathcal{N}/V_d)_{crit}$, where

$$\left(\frac{\mathcal{N}}{V_d}\right)_{crit} \sim C\frac{d+N+1}{a_0^d}\left[\frac{C_N}{C}\frac{2(d-1)}{N}\left(\frac{d+1}{N}+1\right)\right]^{(d+N)/(d+N+1)} \tag{37}$$

For example, we take $d=3$ and $N=7$. In this case,

$$\begin{aligned} C_N &= 5.958744\times10^{-5}\times f\,, \\ C &= \frac{\mathrm{tr}\mathbf{1}}{2}\frac{1}{(4\pi)^{(d+N)/2}}\frac{2\pi^{(N+1)/2}/\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{d+N+2}{2}\right)(d+N+1)} \\ &= 3.139894\times10^{-7}\times f\,, \end{aligned} \tag{38}$$

where f denotes the number of Dirac fields. Then we obtain

$$\left(\frac{\mathcal{N}}{V_d}\right)_{crit} = \frac{f}{a_0^d}\times3.35\times10^{-5}\,. \tag{39}$$

If we consider the initial size of 3-space is the same order as the one of the compact space, Eq. (39) reads the severe constraint to the charge asymmetry.

One can discuss more details (such as including semiclassical instability [22]), by repeating similar analyses of Accetta and Kolb.[23]

6 Summary and discussion

We gave the expressions for the thermodynamic potential of a Dirac fermion gas with the chemical potential in the space $R^d\times S^N$. We mainly pay our

attention to the low-temperature case which leads to the strolls degeneracy of fermi gas. We also gave some examples for applications to a few aspects of Kaluza-Klein theories. We omitted the detailed discussion about the application to the Kaluza-Klein cosmology, which thus will be discussed in another occasion.

The treatment of finite density implicitly assumes the existence of conserved $U(1)$ charge. Superstring theories imply the presence of the gauge field as "primary field," so we may expect some conserved charges. The chemical potentials accompanied with non-abelian charges are also introduced by Haber and Weldon.[11] The generalization to the high-dimensional case will enable us to investigate the finite density effect on the breakdown of primary gauge symmetries as well as compactifications. It may also be interesting to consider the themal property of the gauge field itself [24] in higher dimensions.

On the other hand, it is also interesting to consider the chemical potentials associated to each "Kaluza-Klein charge," which are induced after the compactiacation. This analysis will become necessary when one studies the property of the pyrgons [25] at finite temperature and density. We leave this for future publications.

We need to consider the thermodynamic quantities for boson fields for further study. There is the technical subtlety to calculate the themodynamic potentials in relation to the bosonic field with the chemical potential.[26] But there are also attractive phenomena such as the Bose-Einstein condensation. We will report on these problems generalized to higher-dimensional case elsewhere.[28]

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Appendix

We explain the identity appeared in §§2 and 3 in terms of theta functions. First we observe:

$$\vartheta_3(v|\tau) = \exp(i\pi\tau n^2 + i\pi 2nv)\,, \tag{40}$$

and the well known relation:

$$\vartheta_3\left(\frac{v}{\tau}| - \frac{1}{\tau}\right) = (-i\tau)^{1/2}\exp(i\pi v^2/\tau)\vartheta_3(v|\tau)\,. \tag{41}$$

Using this, (8) can be derived as follows:

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\exp\left\{-t\left(\frac{(2n+1)\pi}{\beta}+i\mu\right)^2\right\}\\ &= \exp\left\{-t\left(i\mu+\frac{\pi}{\beta}\right)^2\right\}\vartheta_3\left(\frac{2i}{\beta}\left(i\mu+\frac{\pi}{\beta}\right)t\,\middle|\,\frac{4\pi}{\beta^2}it\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{4\pi}{\beta^2}t\right)^{-1/2} \vartheta_3\left(\frac{\beta}{2\pi}\left(i\mu+\frac{\pi}{\beta}\right)\middle|\,\frac{\beta^2}{4\pi}\frac{i}{t}\right) \\
&= \left(\frac{\beta^2}{4\pi t}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\beta^2 n^2}{4t}\right) \exp\left\{i\beta\left(i\mu+\frac{\pi}{\beta}\right)n\right\} \\
&= \frac{\beta}{(4\pi)^{1/2}} t^{-1/2}\left[1+2\sum_{n=1}^{\infty}(-1)^n \exp\left(-\frac{\beta^2 n^2}{4t}\right)\cosh(\mu\beta n)\right]. \qquad (42)
\end{aligned}
$$

Next, we show the derivation of (3.2). A similar expansion but for bosons is given by Yoshimura.[27] We consider the case N is odd.

$$
S_F \equiv \sum_{l=0}^{\infty} d_l \, \exp\{-(l+N/2)^2 x\}\,, \quad \text{where} \quad d_l = \frac{\Gamma(l+N)}{l!\,\Gamma(N)}\,. \qquad (43)
$$

We write the degeneracy d_l as

$$
\begin{aligned}
d_l &= \frac{1}{\Gamma(N)}\left[\left(l+\nu+\frac{1}{2}\right)^2-\left(\nu-\frac{1}{2}\right)^2\right]\cdot\left[\left(l+\nu+\frac{1}{2}\right)^2-\left(\nu-\frac{3}{2}\right)^2\right] \\
&\quad \cdots\left[\left(l+\nu+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right] \\
&= \sum_{m=0}^{\nu} C_{\nu m}\left(l+\nu+\frac{1}{2}\right)^{2m}, \qquad \nu=\frac{N-1}{2}\,. \qquad (44)
\end{aligned}
$$

$C_{\nu m}$ are independent of N, and the first two terms are given by

$$
C_{\nu\nu} = \frac{1}{\Gamma(N)}\,, \qquad (45)
$$

$$
C_{\nu\,\nu-1} = -\frac{1}{\Gamma(N)}\sum_{l=1}^{\nu}\left(l-\frac{1}{2}\right)^2 = -\frac{1}{24}\frac{1}{\Gamma(N)}N(N-1)(N-2)\,. \qquad (46)
$$

Since the sum over m vanishes when $l=-1,-2,\cdots,-\nu$, we carry out the summation over l from $l=-\nu$ to ∞. Then, substituting (44) into (43), we obtain

$$
\begin{aligned}
S_F &= \sum_{m=0}^{\nu} C_{\nu m}(-1)^m \frac{d^m}{dx^m}\sum_{l=-\nu}^{\infty}\exp\left[-\left(l+\nu+\frac{1}{2}\right)^2 x\right] \\
&= \sum_{m=0}^{\nu} C_{\nu m}(-1)^m \frac{d^m}{dx^m}\left[\frac{1}{2}\sqrt{\frac{\pi}{x}}+\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}(-1)^n\exp\left(-\frac{\pi^2 n^2}{x}\right)\right]. \qquad (47)
\end{aligned}
$$

Here, we have used the identity derived from (41)

$$
\sum_{l=-\infty}^{\infty}\exp\left[-\left(l+\frac{1}{2}\right)^2 x\right] = \exp\left(-\frac{x}{4}\right)\vartheta_3\left(\frac{ix}{2\pi}\middle|\,\frac{ix}{\pi}\right)
$$

$$
\begin{aligned}
&= \sqrt{\frac{\pi}{x}}\vartheta_3\left(\frac{1}{2}\middle| i\frac{\pi}{x}\right) \\
&= \sqrt{\frac{\pi}{x}}\sum_{n=-\infty}^{\infty}(-1)^n \exp\left(-\frac{\pi^2 n^2}{x}\right) . \qquad (48)
\end{aligned}
$$

From (47) and (44, 45, 46), we find the asymptotic form of S_F,

$$
S_F \overset{x\to 0}{\longrightarrow} \frac{1}{2}\frac{\Gamma(N/2)}{\Gamma(N)}x^{-N/2}\left(1-\frac{1}{12}N(N-1)x+O(x^2)\right) . \qquad (49)
$$

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